

# POINCARÉ SHEAVES ON THE MODULI SPACES OF TORSIONFREE SHEAVES OVER AN IRREDUCIBLE CURVE

USHA N. BHOSLE AND INDRANIL BISWAS

**ABSTRACT.** Let  $Y$  be a geometrically irreducible reduced projective curve defined over  $\mathbb{R}$ . Let  $U_Y(n, d)$  (respectively,  $U'_Y(n, d)$ ) be the moduli space of geometrically stable torsionfree sheaves (respectively, locally free sheaves) on  $Y$  of rank  $n$  and degree  $d$ . Define  $\chi = d + n(1 - \text{genus}(Y))$ , where  $\text{genus}(Y)$  is the arithmetic genus. If  $2n$  is coprime to  $\chi$ , then there is a Poincaré sheaf over  $U_Y(n, d) \times Y$ . If  $2n$  is not coprime to  $\chi$ , then there is no Poincaré sheaf over any nonempty open subset of  $U'_Y(n, d)$ .

## 1. INTRODUCTION

Let  $C$  be a smooth complex projective curve. Let  $U_C(n, d)$  be the moduli space of stable vector bundles over  $C$  of rank  $n$  and degree  $d$ . Assume that  $U_C(n, d)$  is nonempty; this is ensured if  $\text{genus}(C) \geq 2$ . A vector bundle  $\mathcal{E} \rightarrow U_C(n, d) \times C$  is called a *Poincaré vector bundle* if for each point  $z \in U_C(n, d)$ , the vector bundle  $\mathcal{E}|_{\{z\} \times C}$  over  $C$  is in the isomorphism class defined by  $z$ . By a *Poincaré vector bundle* over an open subset  $\mathcal{U} \subset U_C(n, d)$  we will mean a vector bundle  $\mathcal{E} \rightarrow \mathcal{U} \times C$  such that for each point  $z \in \mathcal{U}$ , the vector bundle  $\mathcal{E}|_{\{z\} \times C}$  is in the isomorphism class defined by  $z$ .

If  $n$  is coprime to  $d$ , then there is a Poincaré vector bundle over  $U_C(n, d) \times C$ . If  $n$  and  $d$  are not coprime, then there is no Poincaré vector bundle over any nonempty open subset of  $U_C(n, d)$  [Ra, Theorem 2]. We note that  $n$  and  $d$  are coprime if and only if  $n$  and  $\chi := d - n(\text{genus}(C) - 1)$  are coprime; the Euler characteristic of any vector bundle of rank  $n$  and degree  $d$  over  $C$  is  $\chi$ .

Let  $D$  be a geometrically irreducible smooth projective curve defined over  $\mathbb{R}$ . A vector bundle  $E$  over  $D$  is called geometrically stable if the vector bundle  $E \otimes_{\mathbb{R}} \mathbb{C}$  over  $D \times_{\mathbb{R}} \mathbb{C}$  is stable. Let  $U_D(n, d)$  be the moduli space of geometrically stable vector bundles over  $C$  of rank  $n$  and degree  $d$ . Assume that  $U_D(n, d)$  is nonempty; this is ensured if  $\text{genus}(D) \geq 2$ . If  $D$  has a real point, then there is a Poincaré vector bundle over  $U_D(n, d) \times D$  if and only if  $n$  and  $\chi$  are coprime, where  $\chi$  is defined as above. If  $D$  does not have any real point, then there is a Poincaré vector bundle over  $U_D(n, d) \times D$  if and only if  $2n$  and  $\chi$  are coprime. (See [BiHu].)

Our aim here is to address this question for curves not necessarily smooth.

Let  $Y$  be a geometrically irreducible reduced projective curve defined over the real numbers. A torsionfree sheaf  $V$  on  $Y$  is called geometrically stable if the coherent sheaf

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2000 *Mathematics Subject Classification.* 14F05, 14D20, 14P99.

*Key words and phrases.* Poincaré sheaf, descent condition, stable sheaf, real curve.

on  $Y \times_{\mathbb{R}} \mathbb{C}$  defined by  $V$  is stable. Let  $U_Y(n, d)$  (respectively,  $U'_Y(n, d)$ ) be the moduli space of geometrically stable torsionfree sheaves (respectively, locally free sheaves) on  $Y$  of rank  $n$  and degree  $d$ . We assume that the moduli space  $U_Y(n, d)$  has points defined over  $\mathbb{C}$  (the set of points of  $U_Y(n, d)$  defined over  $\mathbb{R}$  is allowed to be empty); this is ensured if the arithmetic genus of  $Y$  is at least two.

Define  $\chi := d - n(\text{genus}(Y) - 1)$ , which is the Euler characteristic of any sheaf on  $Y$  lying in  $U_Y(n, d)$ .

Assume that  $Y$  does not have any point defined over  $\mathbb{R}$ . We prove the following (see Theorem 3.2):

*If  $2n$  is coprime to  $\chi$ , then there is a Poincaré sheaf over  $U_Y(n, d) \times Y$ .*

*If  $2n$  is not coprime to  $\chi$ , then there is no Poincaré sheaf over any nonempty open subset of  $U'_Y(n, d)$ .*

The proof of Theorem 3.2 given here is different from the one in [BiHu] where it is proved under the assumption that  $Y$  is smooth. The proof in [BiHu] uses the smoothness assumption in an essential way.

## 2. POINCARÉ SHEAF

Let  $X$  be an irreducible reduced projective curve defined over an algebraically closed field  $k$  of characteristic zero. Let  $U(n, d)$  be the moduli space of torsionfree stable sheaves of rank  $n$  and degree  $d$  on  $X$ ; it is a GIT quotient of a Quot scheme  $Q$  of coherent quotient sheaves of  $\mathcal{O}_X^N$  by  $\text{PGL}(N, k)$  [New], [Se]. We assume that  $U(n, d)$  has points defined over  $\mathbb{C}$ ; as mentioned before, the set of points of  $U_Y(n, d)$  defined over  $\mathbb{R}$  may be empty.

Let  $Q'$  denote the set of points of  $Q$  corresponding to stable quotient sheaves. Over  $Q' \times X$ , there is a universal sheaf

$$\mathcal{E} \longrightarrow Q' \times X.$$

Let  $\mathcal{E}_q := \mathcal{E}|_{q \times X}$ .

The points of  $Q' \subset Q$  are identified with the properly stable points for the action of  $\text{PGL}(N, k)$  on  $Q$ . The isotropy subgroup  $G_q$  at  $q$  is isomorphic to  $\text{Aut}(\mathcal{E}_q)$ . Since  $\mathcal{E}_q$  is stable, we have  $G_q = k^*$ ; any  $c \in k^*$  acts on  $\mathcal{E}$  by multiplication with the scalar  $c$ . By a result of Nevins [Nev, p. 2482, Theorem 1.2], the sheaf  $\mathcal{E}$  descends to  $U(n, d) \times X$  if and only if for every  $(q, x) \in Q' \times X$ , the  $\mathcal{O}_{Q' \times X, (q, x)}$ -modules

$$\mathcal{E} \otimes (\mathcal{O}_{Q' \times X}/m_{q, x}) \quad \text{and} \quad \text{Tor}_1^{\mathcal{O}_{Q' \times X}}(\mathcal{E}, \mathcal{O}_{Q' \times X}/m_{q, x})$$

are trivial representations of  $G_q$  (the action of  $G_q$  on  $X$  is taken to be the trivial one); here  $m_{q, x}$  denotes the ideal sheaf of the point  $(q, x) \in Q' \times X$ .

The proof of Proposition 2.1 is straightforward.

**Proposition 2.1.** *Any  $c \in k^*$  acts on the modules*

$$\mathcal{E} \otimes (\mathcal{O}_{Q' \times X}/m_{q, x}) \quad \text{and} \quad \text{Tor}_1^{\mathcal{O}_{Q' \times X}}(\mathcal{E}, \mathcal{O}_{Q' \times X}/m_{q, x})$$

by multiplication by the scalar  $c$ .

- Proposition 2.2.** (1) *There exists a Poincaré sheaf on  $U(n, d) \times X$  if  $n$  and  $d$  are coprime.*  
 (2) *There is no Poincaré sheaf on any open subset of  $U'(n, d)$  if  $n$  and  $d$  are not coprime.*

*Proof.* The first part is proved in [New, Ch. 5, § 7, Theorem 5.12']. We include a proof which will be referred in Section 3.

Let  $\chi := \chi(E_q)$ , where  $q \in Q'$ . Since g.c.d.  $(n, d) = 1$ , there exist integers  $a$  and  $b$  such that  $na + \chi b = -1$ . Let  $\text{Det}(\mathcal{E})$  be the determinant line bundle on  $Q'$  associated to the family  $\mathcal{E} \rightarrow Q' \times X$ . We recall that

$$\text{Det}(\mathcal{E}) = (\det R^0 f_* \mathcal{E}) \otimes (\det R^1 f_* \mathcal{E})^*,$$

where  $f : Q' \times X \rightarrow Q'$  is the projection. Fix a smooth point  $x_0 \in X$ . Define the line bundle on  $Q'$

$$(2.1) \quad \mathcal{L} := (\text{Det}(\mathcal{E}))^b \otimes ((\Lambda^n \mathcal{E}|_{Q' \times x_0})^{\otimes a}).$$

Any  $c \in k^*$  acts on  $\mathcal{L}$  as multiplication by  $c^{-1}$ . Define

$$\mathcal{E}' := \mathcal{E} \otimes p_{Q'}^* \mathcal{L}.$$

Then, by Proposition 2.1, the group  $k^*$  acts trivially on

$$\mathcal{E}' \otimes (\mathcal{O}_{Q' \times X}/m_{q,x}) \quad \text{and} \quad \text{Tor}_1^{\mathcal{O}_{Q' \times X}}(\mathcal{E}', \mathcal{O}_{Q' \times X}/m_{q,x}).$$

By a result of Nevins [Nev, Theorem 1.2], the sheaf  $\mathcal{E}'$  descends to  $U(n, d) \times X$  giving the required Poincaré sheaf.

The second part follows exactly as in [BiHo, Corollary 2.3]. □

### 3. CURVES DEFINED OVER REAL NUMBERS

Let  $Y$  be a geometrically irreducible reduced curve defined over  $\mathbb{R}$ . Let  $U_Y(n, d)$  be the moduli space of geometrically stable torsionfree sheaves on  $Y$  of rank  $n$  and degree  $d$ . We assume that  $U_Y(n, d)$  is nonempty. Let

$$U'_Y(n, d) \subset U_Y(n, d)$$

be the Zariski open subscheme parametrizing the locally free stable sheaves.

**Lemma 3.1.** *Assume that  $Y$  has a smooth real point. Then there is a Poincaré sheaf on  $U_Y(n, d) \times Y$  if  $d$  is coprime to  $n$ . If  $d$  is not coprime to  $n$ , then there is no Poincaré sheaf on any nonempty Zariski open subset of  $U'_Y(n, d)$ .*

*Proof.* Let  $X := Y \times_{\mathbb{R}} \mathbb{C}$  be the complex curve obtained by base change to  $\mathbb{C}$ . We note that the base change  $U_Y(n, d)_{\mathbb{C}} = U_Y(n, d) \times_{\mathbb{R}} \mathbb{C}$  is the moduli space  $U_X(n, d)$  of stable torsionfree sheaves on  $X$  of rank  $n$  and degree  $d$ . Similarly,  $U'_Y(n, d)_{\mathbb{C}} = U'_Y(n, d) \times_{\mathbb{R}} \mathbb{C}$  is the moduli space  $U'_X(n, d)$  of stable vector bundles on  $X$  of rank  $n$  and degree  $d$ .

If  $\mathcal{E} \rightarrow \mathcal{U} \times Y$  is a Poincaré sheaf, where  $\mathcal{U} \subset U'_Y(n, d)$  is a nonempty Zariski open subset, then  $\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}$  is a Poincaré sheaf on  $(\mathcal{U} \times_{\mathbb{R}} \mathbb{C}) \times X$ . In that case, Proposition 2.2(2) says that  $d$  is coprime to  $n$ .

Conversely, if  $d$  is coprime to  $n$ , taking the point  $x_0$  in the proof of Proposition 2.2(1) to be a real point we see that the Poincaré sheaf constructed in the proof of Proposition 2.2 is defined over  $\mathbb{R}$ .  $\square$

Henceforth, we assume that  $Y$  does not have any real point.

As in Section 2, define

$$(3.1) \quad \chi := d + n(1 - \text{genus}(Y)),$$

where  $\text{genus}(Y)$  is the arithmetic genus; note that  $\chi = \chi(E)$  for any  $E \in U_Y(n, d)$ .

**Theorem 3.2.** *Assume that  $Y$  does not have any point defined over  $\mathbb{R}$ .*

- (1) *If  $2n$  is coprime to  $\chi$ , then there is a Poincaré sheaf over  $U_Y(n, d) \times Y$ .*
- (2) *If  $2n$  is not coprime to  $\chi$ , then there is no Poincaré sheaf over any nonempty open subset of  $U'_Y(n, d)$ .*

*Proof.* For a variety  $Z$  defined over  $\mathbb{C}$ , let

$$\overline{Z} := (Z(\mathbb{C}), \overline{\mathcal{O}}_Z)$$

be the complex conjugate variety of  $Z$ . As in the proof of Lemma 3.1, define  $X := Y \times_{\mathbb{R}} \mathbb{C}$ . Let

$$(3.2) \quad \sigma : X \rightarrow \overline{X}$$

be the natural isomorphism obtained from the fact that  $X$  is the base change of  $Y$  to  $\mathbb{C}$ . The composition

$$(3.3) \quad X \xrightarrow{\sigma} \overline{X} \xrightarrow{\overline{\sigma}} \overline{\overline{X}} = X$$

is the identity map of  $X$ .

First assume that  $2n$  is coprime to  $\chi$ . Fix a smooth effective real divisor  $D$  on  $Y$  of degree two. We note that such divisors exist; they are in bijective correspondence with the pairs of smooth points of  $X$  of the form  $\{x, \sigma(x)\}$ , where  $\sigma$  is the map in (3.2).

That there is a Poincaré sheaf over  $U_Y(n, d) \times Y$  can be shown exactly as done in the proof of the first part of Proposition 2.2. Instead of the smooth point  $x_0$  in the proof of Proposition 2.2, we take the above divisor  $D$ . More precisely, replace  $\Lambda^n \mathcal{E}|_{Q' \times x_0}$  in (2.1) by the line bundle  $\Lambda^{2n} \mathcal{E}|_{Q' \times D}$ ; note that  $\mathcal{E}|_{Q' \times D}$  is a vector bundle of rank  $2n$  over  $Q'$ . Since  $2n$  is coprime to  $\chi$  the rest of the argument remains unchanged.

Now assume that  $\chi$  is not coprime to  $2n$ .

The Euler characteristic  $\chi$  is coprime to  $n$  if and only if  $d$  is coprime to  $n$ . Therefore, if  $\chi$  is not coprime to  $n$ , then from Proposition 2.2 we know that there is no Poincaré sheaf on any nonempty Zariski open subset of  $U'_Y(n, d)$ ; note that any Poincaré sheaf on  $\mathcal{U} \times Y$  defines a Poincaré sheaf on  $(\mathcal{U} \times_{\mathbb{R}} \mathbb{C}) \times X$  by base change to  $\mathbb{C}$ .

Consequently, we assume  $\chi$  is coprime to  $n$ .

Since  $\chi$  is not coprime to  $2n$ , we conclude that  $\chi$  is even. Therefore,  $n$  is odd because  $\chi$  is coprime to  $n$ . Define

$$(3.4) \quad b_0 := (n-1)/2 \in \mathbb{Z}.$$

There is an integer  $\delta$  and a real point

$$L_0 \in \text{Pic}^\delta(Y)$$

such that  $L_0$  is not a line bundle over  $Y$  [BK, p. 226, Corollary 2] (see [GH, p. 159, Proposition 2.2(2)] for smooth  $Y$ ). Since there are line bundles on  $Y$  of degree two (recall that there is a smooth real divisor of degree two), for any integer  $m$ , we have the real point

$$L_0 \otimes \xi^{\otimes m} \in \text{Pic}^{\delta+2m}(Y),$$

where  $\xi \rightarrow Y$  is a line bundle of degree two, which is not a line bundle over  $Y$ .

We will consider  $L_0$  as a line bundle over  $X$ , because it defines a point of  $\text{Pic}^{\delta+2m}(X)$ . Note that  $H^i(X, L_0)$  is of even dimension because it has a quaternionic structure. Hence  $\chi(L_0)$  is even. Therefore, for any even integer  $b$ , there is a real point of  $\text{Pic}^{b-1+\text{genus}(Y)}(Y)$  which is not a line bundle over  $Y$ ; note that the Euler characteristic is  $b$ .

We may assume that the degree  $d$  is sufficiently large positive by tensoring with the line bundle  $\mathcal{O}_Y(aD)$ , where  $D$  as before is a real smooth effective divisor of degree two, and  $a \in \mathbb{N}$ . Note that  $\chi$  is also sufficiently large positive because  $\chi = d - n(\text{genus}(Y) - 1)$ .

We noted earlier that  $n$  is odd, and  $\chi$  is even. Hence

$$(3.5) \quad \chi + n(\text{genus}(Y) - 1) + 1 - \text{genus}(Y) \equiv 0 \pmod{2}.$$

We also noted above that for any even integer  $b$ , there is a real point of  $\text{Pic}^{b-1+\text{genus}(Y)}(Y)$  which is not a line bundle over  $Y$ . Hence from (3.5) we conclude that there is a real point

$$(3.6) \quad L \in \text{Pic}^{\chi+n(\text{genus}(Y)-1)}(Y) = \text{Pic}^d(Y)$$

which is not a real line bundle over  $Y$ . Fix such a point  $L$ .

The line bundle over  $X$  (respectively,  $\overline{X}$ ) (see (3.3)) corresponding to  $L$  in (3.6) will be denoted by  $L$  (respectively,  $\overline{L}$ ). Since  $L$  in (3.6) is a real point, but not a line bundle on  $Y$ , there is a unique isomorphism

$$(3.7) \quad \eta : L \rightarrow \sigma^* \overline{L}$$

such that  $(\sigma^* \overline{\eta}) \circ \eta = -\text{Id}_L$ . (see (3.3)).

Let

$$(3.8) \quad T : \mathcal{O}_X \oplus \mathcal{O}_X \rightarrow \mathcal{O}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}}$$

be the isomorphism defined by  $(f_1, f_2) \mapsto (-f_2 \circ \sigma^{-1}, f_1 \circ \sigma^{-1})$ . We note that

$$(\sigma^* \overline{T}) \circ T = -\text{Id}_{\mathcal{O}_X^{\oplus 2}}.$$

Consider  $b_0$  defined in (3.4). Elements of  $H^1(X, L^\vee \otimes (\mathcal{O}_X^{\oplus 2}))^{\oplus b_0}$  parametrize extensions of the form

$$0 \longrightarrow (\mathcal{O}_X^{\oplus 2})^{\oplus b_0} \longrightarrow V \longrightarrow L \longrightarrow 0,$$

where  $L$  is the above line bundle on  $X$ . There is a universal short exact sequence

$$(3.9) \quad 0 \longrightarrow (\mathcal{O}_{X \times \mathbb{A}}^{\oplus 2})^{\oplus b_0} \longrightarrow \mathcal{V} \longrightarrow p_1^* L \longrightarrow 0,$$

where  $\mathbb{A} := H^1(X, L^\vee \otimes (\mathcal{O}_X^{\oplus 2}))^{\oplus b_0}$ , and  $p_1$  is the projection of  $X \times \mathbb{A}$  to  $X$ . Since  $d$  is sufficiently large, all stable vector bundles over  $X$  of rank  $n$  and determinant  $L$  occur in the family  $\mathcal{V}$  in (3.9). Let

$$(3.10) \quad \mathcal{S} \subset \mathbb{A}$$

be the locus of stable bundles for the family in (3.9); from the openness of the stability condition (see [Ma]) it follows that  $\mathcal{S}$  is a Zariski open subset. Let  $U'_X(n, L)$  be the moduli space of stable vector bundles  $E$  over  $X$  of rank  $n$  with  $\bigwedge^n E = L$ . Let

$$(3.11) \quad \Phi : \mathcal{S} \longrightarrow U'_X(n, L)$$

be the surjective morphism representing the family  $\mathcal{V}$  in (3.9).

Let

$$(3.12) \quad \psi : U'_X(n, L) \longrightarrow U'_X(n, L)$$

be the bijection defined by  $E \mapsto \sigma^* \overline{E}$ , where  $\sigma$  is the isomorphism in (3.2), and  $\overline{E}$  is the vector bundle over  $\overline{X}$  corresponding to the vector bundle  $E$  over  $X$ . It should be clarified that  $\psi$  is not algebraic, it is not even holomorphic, but anti-holomorphic. Let  $\overline{U'_X(n, L)}$  be the variety obtained from  $U'_X(n, L)$  using the automorphism of the field  $\mathbb{C}$  defined by  $z \mapsto \overline{z}$ . Therefore, the complex points of  $\overline{U'_X(n, L)}$  are in bijective correspondence with the complex points of  $U'_X(n, L)$ . Using this bijection, if we consider  $\psi$  in (3.12) as a map  $U'_X(n, L) \longrightarrow \overline{U'_X(n, L)}$ , then this map is an algebraic isomorphism.

The isomorphism  $\psi$  in (3.12) is clearly involutive. This  $\psi$  defines a real structure on the complex variety  $U'_X(n, L)$ . The corresponding variety over  $\mathbb{R}$  is the moduli space  $U'_Y(n, L) := \det^{-1}(L)$ , where

$$(3.13) \quad \det : U'(n, d) \longrightarrow \text{Pic}^d(Y)$$

is the morphism defined by  $E \mapsto \bigwedge^n E$ , and  $L$  is the point in (3.6).

Let

$$(3.14) \quad \theta : H^1(X, L^\vee \otimes (\mathcal{O}_X^{\oplus 2}))^{\oplus b_0} \longrightarrow H^1(X, L^\vee \otimes (\mathcal{O}_X^{\oplus 2}))^{\oplus b_0}$$

be the conjugate linear involution constructed using  $\eta$  and  $T$  defined in (3.7) and (3.8) respectively. The subset  $\mathcal{S}$  in (3.10) is preserved by  $\theta$ , and

$$(3.15) \quad \psi \circ \Phi = \Phi \circ \theta,$$

where  $\Phi$  and  $\psi$  are constructed in (3.11) and (3.12) respectively. Therefore, the morphism  $\Phi$  is defined over  $\mathbb{R}$ .

The fixed point locus

$$(3.16) \quad \mathbb{A}^\theta \subset \mathbb{A}$$

for  $\theta$  is a  $\mathbb{R}$ -linear subspace such that the natural homomorphism

$$\mathbb{A}^\theta \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow H^1(X, L^\vee \otimes (\mathcal{O}_X^{\oplus 2})^{\oplus b_0})$$

is an isomorphism; hence  $\mathbb{A}^\theta$  is Zariski dense in  $\mathbb{A}$ .

Let

$$(3.17) \quad \mathcal{W} \longrightarrow \mathcal{U} \times Y$$

be a Poincaré sheaf, where  $\mathcal{U}$  is a nonempty Zariski open subset of  $U'_Y(n, d)$ . The morphism  $\det$  in (3.13) is an open smooth surjective morphism, hence the image  $\det(\mathcal{U})$  is a nonempty Zariski open subset of  $\text{Pic}^d(Y)$ . We noted earlier that there is a real point of  $\text{Pic}^d(Y)$  which are not a line bundle on  $Y$  (see (3.6)), and also it is known that any (nonempty) connected component of the locus of real points in a smooth quasiprojective variety is Zariski dense. Hence the set of real points of  $\text{Pic}^d(Y)$  which are not line bundles on  $Y$  is Zariski dense in  $\text{Pic}^d(Y)$ . In particular, this set intersects the Zariski open subset  $\det(\mathcal{U})$ , where  $\mathcal{U}$  is the open subset in (3.17). Therefore, we may take the chosen point  $L$  in (3.6) to be inside  $\det(\mathcal{U})$ . Hence we assume that

$$L \in \det(\mathcal{U}).$$

Since  $\mathcal{S}$  in (3.10) is a nonempty Zariski open subset, and  $\mathbb{A}^\theta$  defined in (3.16) is Zariski dense, we know that  $\mathcal{S} \cap \mathbb{A}^\theta$  is Zariski dense in  $\mathbb{A}$ . Take any point

$$(3.18) \quad z \in \mathcal{S} \cap \mathbb{A}^\theta$$

such that the corresponding vector bundle

$$(3.19) \quad \mathcal{V}_z := \mathcal{V}|_{\{z\} \times X}$$

lies in the open subset  $\mathcal{U}$  in (3.17). Since  $z \in \mathbb{A}^\theta$ , from (3.15) we know that  $\psi(\Phi(z)) = \Phi(z)$ , where  $\psi$  and  $\Phi$  are defined in (3.12) and (3.11) respectively. Therefore, there is an isomorphism

$$(3.20) \quad \beta : \mathcal{V}_z \longrightarrow \sigma^* \overline{\mathcal{V}_z},$$

constructed using  $\sigma$  and  $T$  defined in (3.2) and (3.8) respectively, such that

$$(3.21) \quad (\sigma^* \overline{\beta}) \circ \beta = -\text{Id}_{\mathcal{V}_z}.$$

This isomorphism  $\beta$  fits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}_X^{\oplus 2})^{\oplus b_0} & \longrightarrow & \mathcal{V}_z & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow \beta & & \downarrow \eta & & \\ 0 & \longrightarrow & \sigma^*(\mathcal{O}_X^{\oplus 2})^{\oplus b_0} = (\mathcal{O}_X^{\oplus 2})^{\oplus b_0} & \longrightarrow & \sigma^* \overline{\mathcal{V}_z} & \longrightarrow & \sigma^* \overline{L} & \longrightarrow & 0 \end{array}$$

where the horizontal exact sequences are as in (3.9), and the maps  $T$  and  $\eta$  are constructed in (3.8) and (3.7) respectively.

Let  $\underline{z} \in U'(n, d)$  be the point corresponding to the vector bundle  $\mathcal{V}_z$  in (3.19). Restricting the Poincaré bundle  $\mathcal{W}$  to  $\{\underline{z}\} \times Y$  we get a vector bundle over  $Y$  which is represented by this point  $\underline{z}$  of the moduli space.

For any geometrically stable vector bundle  $W_0$  over  $Y$ , the group of all automorphisms of the corresponding vector bundle  $W$  over  $X$  is the group of nonzero complex numbers. There is a natural isomorphism

$$\gamma : W \longrightarrow \sigma^* \overline{W}$$

such that  $(\sigma^* \overline{\gamma}) \circ \gamma = \text{Id}_W$ . Since any other isomorphism  $W \longrightarrow \sigma^* \overline{W}$  must be of the form  $c \cdot \gamma$ , where  $c \in \mathbb{C}^*$  (recall that the automorphisms of  $W$  are the nonzero scalars), we conclude that there is no isomorphism

$$\phi : W \longrightarrow \sigma^* \overline{W}$$

such that  $(\sigma^* \overline{\phi}) \circ \phi = -\text{Id}_W$  (there is no complex number such that  $\overline{c}c = -1$ ).

But the vector bundle  $\mathcal{W}|_{\{\underline{z}\} \times Y}$  in (3.17) contradicts the above observation that it is not possible to have simultaneously isomorphisms  $\gamma$  and  $\phi$  of the above type. From this contradiction we conclude that there is no Poincaré sheaf over  $\mathcal{U} \times Y$ . This completes the proof of the theorem.  $\square$

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

*E-mail address:* usha@math.tifr.res.in

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

*E-mail address:* indranil@math.tifr.res.in